MODELING AN EXTERNALLY PRESSURIZED COAXIAL PIPE SYSTEM
USING ELASTIC COUPLED BEAM THEORY

BY

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THESIS

Submitted in partial fulfillment of the requirements for
the degree of Master of Science in Mechanical Engineering
in the Graduate School of
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Accepted in partial fulfillment of the requirements for the degree of Master of Science in Mechanical Engineering in the Graduate School of Binghamton University State University of New York 2010

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ABSTRACT

It is well known that an external symmetrically pressurized cylindrical shell can be modeled using beam theory. The shell theory derived governing differential equation describing the static inward pinch of the cylindrical shell is known to be of the same form as the Euler-Bernoulli equation describing the small static displacement of a beam on a Winkler Elastic Foundation. The idea behind this research was to determine if it is feasible to model two coaxial pipes, with a variable material existing between them, and a uniform external pressure distribution, using methods of coupled elastic beam theory. Through a differential calculus derivation, it was determined that the equations governing the static radial deflection of the cylindrical shell system are similar to that of the static coupled beam problem, but contained two extra terms. The solution to the single cylindrical shell problem is special case of the newly derived coupled cylindrical shell problem.

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DEDICATION

To family, friends, colleagues, and students.

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CHAPTER 1: Problem Statement

1.1 Literature Review

The idea of linear elastic layers has been a valuable tool used in industry. These layers are versatile, and not prone to any one specific type of modeling. The elastic layer itself can act as an elastic foundation, while it can also be used as a means of coupling two or more bodies. Several successful key layer models include, but are not limited to, railroad rails, highway slabs, and printed circuit boards. While a countless number of models have already been implemented to exercise this tool, there are still a numerous amount problems that have yet to be solved.

The beam on an elastic foundation problem was first developed by Winkler, in 1867, when there was a strong need to analyze the deflection of a railroad track. Winkler's idea of a linear reacting elastic foundation still drives most analyses performed today. Much later, the idea of the elastically coupled beam was introduced. This problem was first solved by Suhir (1988) for the case of end moment loading only. Then later it was again solved by Engel, Caletka, and Palmer (1990, 1991, 1993) using a central point loading. The coupled beam work done by Engel has since been modified and corrected by Pitarresi (1999, 2001).

Cook states and proves a relationship between two seemingly unrelated topics. The governing differential equation describing a cylindrical shell's static radial deformation is of the same form as the equation describing the static deflection of a beam on a Winkler linear elastic foundation (Cook, 1985). Knowing that this relationship exists, means there is more than one way to model this particular system.

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Lastly, to this day, no literature has attempted to solve a problem involving an elastic coupled cylindrical shell system. The idea is remarkably similar to the elastic coupled beam problem in that two bodies are being joined by a linear Winkler elastic layer. The difference is in the dimensioning of the problems. The beam problem takes a one dimensional form, while the shell problem spans two dimensions. Using a similar methodology which was used to derive the coupled beam problem, it is possible obtain a solution for the coupled elastic cylindrical shell system.

1.2 Problem Statement

The proposed question in this Thesis is the following. Does the coupled cylindrical shell system problem yield exact, or perhaps similar governing coupled differential equations to that of the coupled beam problem? Are there main assumptions that need to be enforced in order to derive these governing equations of the shell problem? If the equations are exact, then what are the defining constant coefficients and their meanings? If the equations differ, are there a set of conditions that can be applied to make the equations of the same form?

There are a countless number of structural applications that involve coupled cylindrical shells in various disciplines of engineering. Knowing how a coupled shell stretches and bends to a particular symmetric loading can help engineers get a general basis on allowable dimensions for a system and can be a means of start for weight optimization.

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1.3 Proposed Method

Throughout this Thesis, all derivations use differential (infinitesimal) calculus to setup and derive each system's governing differential equations. The basic methodology is as follows. A small differential element of each particular system undergoing a load is analyzed. Once all exposed internal reactions, and changes in internal reaction are shown in the differential free body diagram, equilibrium can then be enforced. Newton's equilibrium laws are used to obtain the differential equations in terms of internal reactions and external loadings. Next, constitutive, and compatibility laws are utilized to relate the system's geometry and internal loads to its lateral change in displacement.

Once the governing differential equations of each system are obtained, three methods will be used to obtain their solutions. The first methodology is obtaining an analytical solution via basic differential equation principles. For more complicated systems of differential equations, a Numerical approach will be taken using MATLAB. Lastly an ANSYS finite element analysis will be used to model and check the derived solutions within the thesis.

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CHAPTER 2: Background

2.1 Beam on Elastic Foundation

Analysis of beams lying on an elastic medium was first done by Winkler in 1867 (Cook, 1985). When it was discovered that railroad track roadbeds resist deformation, it became Winkler's job to model the phenomenon. Today this developed model is known as the Winkler Elastic Foundation. The Winkler model has been shown to be an effective way to model "railroad rails, piers supported by piling and loaded by horizontal force, networks of beams in floor systems, highway slabs, and structures that float" (Cook, 1985).

The Winkler Model is the simplest elastic foundation representation model still around and used today. Like Hooke's law it presumes a linear force-displacement relationship with a given stiffness, or in the case of an elastic medium, pressure foundation modulus (k_o, N/m^2/m). The foundation has units of pressure per displacement and can be visualized as a bed of infinite springs. For the case of beams with uniform cross section, the foundation modulus can be thought to resist deflection with a distributed load rather than a pressure. In this paper, the following equation denotes the foundation modulus for a uniform cross section beam.

$$k = bk_o \quad \textbf{(Eq. 2.1.1)}$$

In Equation 2.1.1, k is the new foundation modulus (N/m/m), and b is simply the base measurement of the beam (m). When k is multiplied by a unit displacement, the elastic foundation acts as a distributed load opposing the beams lateral displacement.

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It should be noted that the Winkler model assumes the beam never loses contact with the foundation, meaning it is allowed to have a negative pressure if beam displaces in the opposite direction of the foundation. The model works well when studying how a uniquely loaded beam deforms on a foundation. However, if one is interested in learning about the physical characteristics and affects of the foundation itself, a more complex model would be necessary. The Winkler elastic foundation displaces only where the beam itself deflects. A true foundation would displace both under the beam and adjacent to the beam (Cook, 1985).

2.2 Development of the Equation for Beam on Elastic Foundation

For the case of this Thesis, the interest is geared towards the static deflection equation for a beam on an elastic foundation. This equation takes the same form as the symmetrically loaded cylindrical shell.

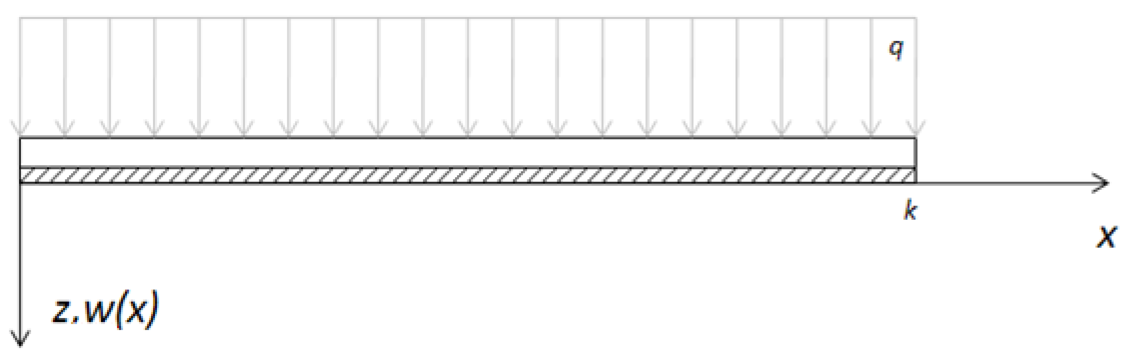

Figure 1. Beam on Elastic Foundation Model

Figure 1 shows the standard coordinate system setup for the uniform beam on an elastic foundation carrying a distributed load $q(x)$. The beam lies on the x-axis and displaces along the z-axis as a function of $w(x)$. For now we are only concerned with the differential equation describing the beams static displacement, therefore no

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boundary conditions are needed for the setup of the derivation. Later in Section 4.1, a specific case scenario with applied boundary conditions will be solved for.

In order to obtain the differential equation describing the beams displacement, a small differential element, with exposed internal forces and moments, will be taken. Laws of equilibrium and compatibility will then be exercised to achieve the differential equation in terms of just the dependent variable *w(x)*, the deflection of the beam.

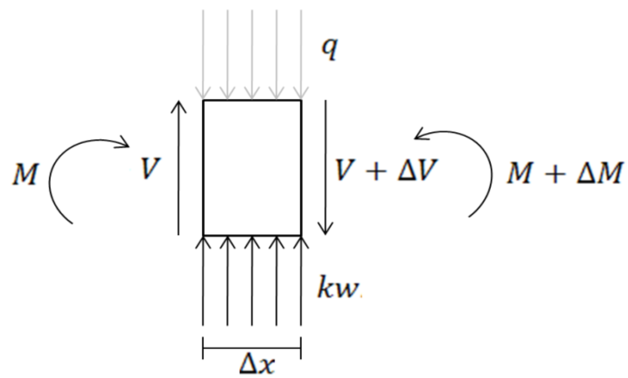

Figure 2. Differential Element of Figure 1 Beam

The differential element in Figure 2 has a uniform base thickness *b*, and a length Δx. A load *q(x)* is uniformly distributed over the top of element, and an elastic foundation spans the bottom, resisting the beams displacement with distributed load *kw(x).* A shear force, *V,* and couple moment, *M*, are shown in their positive orientation. Both of these internal components vary in the x-direction by some $\Delta V \ and \ \Delta M$.

First, by invoking equilibrium in the z-direction, the following equation is obtained.

$$\sum F_Z = q\Delta x - kw\Delta x + V + V + \Delta V = 0 \quad \textbf{(Eq. 2.2.1)}$$

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Rearranging terms and assuming the shear force varies linearly across the infinitesimal piece, Equation 2.2.1 becomes

$$\frac{dV}{dx} = kw - q. \quad \textbf{(Eq. 2.2.2)}$$

Equation 2.2.2 states the change in shear along the x-direction is equal to negative summation of distributive loads (forces per unit length) acting laterally on the beam.

Next, by invoking a moment equilibrium, summing the moments about the center of the element, the following equation is obtained.

$$\sum M_{center} = -M + M + \Delta M - \frac{1}{2}\Delta x V - \frac{1}{2}\Delta x V - \frac{1}{2}\Delta x \Delta V = 0 \quad \textbf{(Eq. 2.2.3)}$$

Rearranging terms, Eq. 2.2.3 becomes,

$$\frac{dM}{dx} = V. \quad \textbf{(Eq. 2.2.4)}$$

Equation 2.2.4 states the derivative of the moment with respect to the x-coordinate is equal to the shear at any point on the beam. Knowing beams only support lateral loads, the last planar equilibrium equation is automatically satisfied and serves no purpose in the derivation. Subbing Equation 1.2.4 into Equation results in the following equation

$$-\frac{d^2M}{dx^2} + kw = q. \quad \textbf{(Eq. 2.2.5)}$$

Lastly, a final compatibility relationship relating the moment at any place on the beam, to the beams deflection needs to be obtained. To do this it is necessary to first relate the radius of curvature to the normal strain in the beam.

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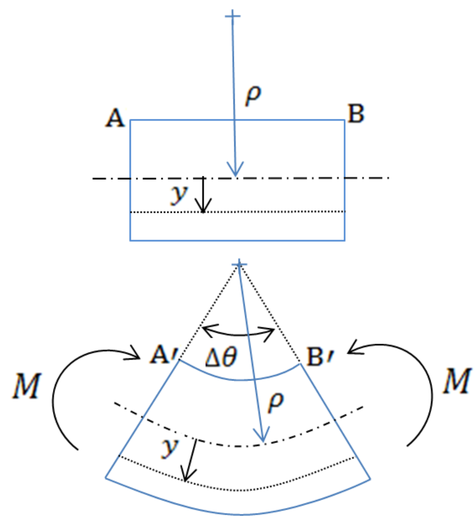

Figure 3. Beam in Positive Bending

Figure 3 shows a differential element before and after positive pure bending. Before bending the length of the beam remains the same length *AB* for any distance y from the neutral axis. After undergoing pure bending, *A'B'* becomes shortened above the neutral axis and longer beneath the neutral axis. The following equation shows the stain, ε_x relationship.

$$\varepsilon_x = \frac{\overline{\mathrm{AB}} - \overline{\mathrm{A'B'}}}{\overline{\mathrm{AB}}} = \frac{\rho\Delta\theta - (\rho + y)\Delta\theta}{\rho\Delta\theta} = -\frac{y}{\rho} \quad \textbf{(Eq. 2.2.6)}$$

Equation 1.2.6 shows the strain ε_x in terms of y, the distance from the neutral axis and ρ, the radius of curvature. It is worth noting that, with no torsion in the beam, the strain remains constant through the cross section of the beam.

Next, applying Hooke's law in one dimension to Equation 1.2.6 yields the following equation.

$$\varepsilon_x = \frac{\sigma_x}{E} = -\frac{y}{\rho} \quad \textbf{(Eq. 2.2.7)}$$

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Equation 1.2.7 relates the axial stress at some y to the radius of curvature of the beam, ρ and the beams modulus of elasticity, E.

Taking the summation of moments on the neutral axis at the cut of the differential element, results in the following equation.

$$\sum M_{neutral} = M(x) = -\int(\sigma_x y \, dA) \quad \textbf{(Eq. 2.2.8)}$$

Rearranging Equation 2.2.8, along with subbing in Equation 2.2.7 yields the following equation.

$$M = \int \frac{Ey^2}{\rho} \, dA = \frac{E}{\rho}\int y^2 dA = \frac{EI}{\rho} = EI\kappa_x \quad \textbf{(Eq. 2.2.9)}$$

Equation 2.2.9 is the final moment curvature relationship relating the moment anywhere on the beam to the beams sectional modulus and curvature at that point.

The curvature of a single direction plane curve can be expressed in terms of the lateral displacement of the curve with respect to its original shape.

$$\kappa_x = -\frac{\frac{d^2w}{dx^2}}{\left(1+\left(\frac{dw}{dx}\right)^2\right)^{\frac{3}{2}}} \approx -\frac{d^2w}{dx^2} \quad for \quad \frac{dw}{dx} \ll 1 \quad \textbf{(Eq. 2.2.10)}$$

For small curvature Equation 2.2.10 is an adequate approximation describing the single direction curvature of a plane in terms of the change in displacement *w(x).*

Substituting Equation 2.2.10 into Equation 2.2.9 returns the final compatibility equation needed to finish the derivation of the beam on an elastic foundation.

$$M = -EI\frac{d^2w}{dx^2} \quad \textbf{(Eq. 2.2.11)}$$

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Substituting Equation 2.2.11 into Equation 2.2.5 gives the final single differential equation relationship describing the displacement of the beam *w(x)*, with uniform distributive load *q(x)*, and elastic foundation with foundation modulus *k*.

$$EI\frac{d^4w}{dx^4} + kw = q \quad \textbf{(Eq. 2.2.12)}$$

Ordinary differential Equation 2.2.13 takes a forth-order, constant coefficient form, hosting *w(x)* as its dependent variable.

2.3 Beam on Elastic Foundation Analytical Solution

Having constant coefficients, Equation 2.2.12 can be easily solved analytically. First, it is assumed that displacement function w(x), takes the following exponential form

$$w(x) = e^{\lambda x}, \quad \textbf{(Eq. 2.3.1)}$$

where λ is an unknown constant. Next, Equation 2.3.1 is substituted into the homogeneous form of Equation 2.2.12 results in the following equation.

$$(EI\lambda^4 + k)e^{\lambda x} = 0 \quad \textbf{(Eq. 2.3.2)}$$

Equation 2.3.2 yields the differential equation's characteristic equation, which has four complex roots. For convenient the following notation will be introduced (Cook, 1985).

$$\beta = \left(\frac{k}{4EI}\right)^{\frac{1}{4}} \quad \textbf{(Eq. 2.3.3)}$$

Using this notation, the roots for characteristic Equation 2.3.2 can be written as the following.

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$$\lambda_{1,2} = \beta \pm i\beta \qquad \lambda_{3,4} = -\beta \pm i\beta$$

Through arguments of differential equation theory, the analytical homogeneous solution to the deflection, *w(x),* of a beam on an elastic foundation can be written as

$$w_h(x) = e^{\beta x}(C_1 Sin[\beta x] + C_2 Cos[\beta x]) + e^{-\beta x}(C_3 Sin[\beta x] + C_4 Cos[\beta x]). \quad \textbf{(Eq. 2.3.4)}$$

Equation 2.3.4 contains unknown constants C_1 thru C_4, which are found by satisfying known boundary conditions, and constant β, which describes the beam and foundation's physical properties. The final solution for a Winkler foundation beam undergoing a uniform distributed load q, is simply

$$w(x) = e^{\beta x}(C_1 Sin[\beta x] + C_2 Cos[\beta x]) + e^{-\beta x}(C_3 Sin[\beta x] + C_4 Cos[\beta x]) + \frac{q}{k}, \quad \textbf{(Eq. 2.3.4)}$$

where $\frac{q}{k}$ is the particular solution to Equation 2.2.12.

2.4 Cylindrical Shell with External Uniform Pressure

Several engineering problems are concerned with how a pipe deforms when a uniform pressure is introduced to its outer surface. This inward pinch of the pipe's mid-thickness neutral axis can be described using the theory of shell structures. The derivation of the governing differential equation makes use of several key assumptions to help simplify the equation's final form. The first assumption is that the pipe is made of an isotropic, linear elastic material. Next, the radius of the pipe needs to be significantly greater than its thickness. Lastly, the deformation of the pipe needs to remain small so small angle approximation and curvature laws can apply.

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2.5 Development of the Equation for a Cylindrical Shell External Pressure

The assumptions stated in Section 2.4 can be applied to most thin wall piping, making shell theory a plausible methodology to describe the pipes deformation.

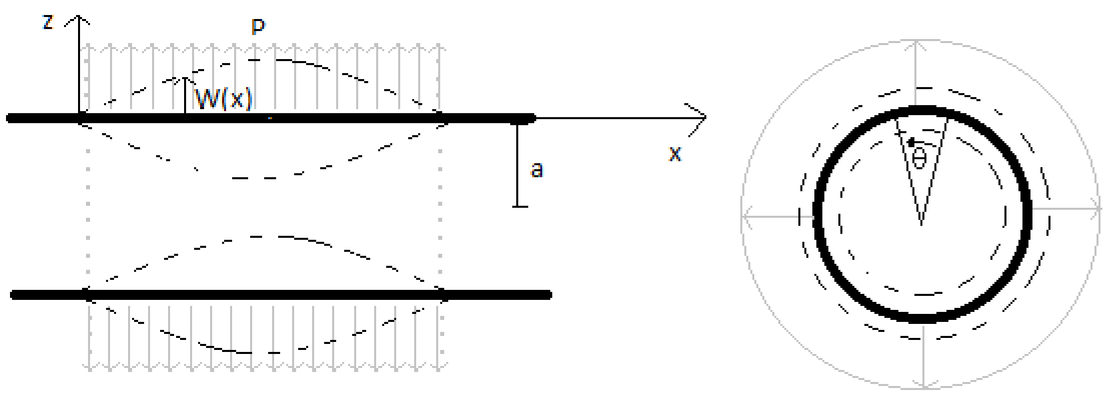

Figure 4. Pipe Coordinate System Setup

Figure 4 shows the general coordinate system that will be used throughout the entire derivation. Through uniform radial pressure loading *P*, the radius to the neutral axis remains uniform for the entire pipe. The change of radius is designated as *w(x)*, which defined as the deformation of the neutral axis radius with respect to the pipe's original none loaded radius, *a*. The dotted lines show a greatly exaggerated positive and negative pressure deformation, to clearly illustrate the coordinate system. The x-axis runs through the non-deformed neutral surface along the length of the pipe. The z-axis runs perpendicular to the non-deformed neutral surface, positive notation running away from the center of the cylindrical shell. Lastly, angle coordinate θ, is introduced for the derivation, completing the orthogonal coordinate system. The θ coordinate does not appear in the final governing differential equation. Next, by taking a

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differential piece of the shell's neutral surface, the following stress resultants become exposed.

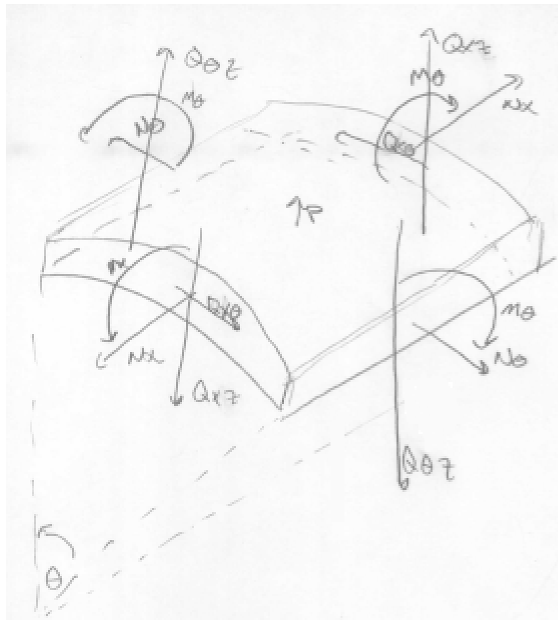

Figure 5. Differential Element of Figure 4 Model

Figure 5 shows all stress resultants with their positive sign convention notation (top surface assumed to be in tension). All internal resultants have units of reaction per length. The resultants acting on the x face are, normal stress, N_x, tangential shear, $Q_{x\theta}$, transverse shear, Q_{xz} and finally bending moment, M_x. The pipe being analyzed in this section has neither axial nor torsional loading, meaning the N_x, $Q_{x\theta}$, and $Q_{\theta Z}$ resultants are zero for this specific scenario. The resultants on the θ face are, normal stress, N_θ, tangential shear $Q_{\theta z}$, and moment M_θ. Due to the radial loaded symmetry of the problem both $Q_{\theta z}$ and M_θ also go to zero.

The same methodology used in Section 2.2 will be used when deriving the cylindrical shells governing differential equation. First, equilibrium will be invoke in the z-direction.

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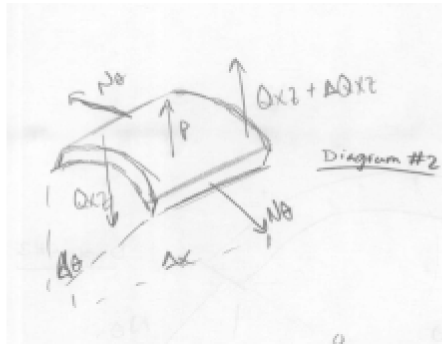

Figure 6. Differential Element, z direction, Non-zero Forces/Lengths

Figure 6 shows the necessary stress resultants and their corresponding change in values across the differential piece, required to write the z-direction equilibrium equation. The tangential resultant and the pressure loading's replaced point force, both lay in the z-direction. The circumferential normal resultant's z-component can be written knowing the following.

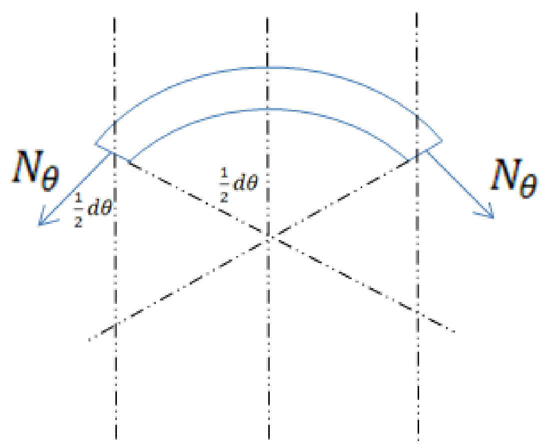

Figure 7. Circumferential Normal Force Calculation

N_θ like M_θ does not change throughout the entire circumference of the shell because of the symmetry induced in the problem. Through small angle approximation the circumferential force z-component can be written as the summation of the two

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$N_\theta dx\frac{1}{2}d\theta$ face resultant forces. Writing the equilibrium equation for the forces in the z-direction, gives the following expression.

$$\sum F_Z = \Delta Q_x\, a\Delta\theta + Q\Delta xa\Delta\theta - Q\Delta xa\Delta\theta - N_\theta\Delta x\,\Delta\theta + P\Delta xa\Delta\theta = 0 \quad \textbf{(Eq. 2.5.1)}$$

Next, dividing through by $\Delta xa\Delta\theta$, assuming results vary linearly for small segment, and rearranging terms in Equation 2.5.1 yields the following differential equation.

$$\frac{dQ_x}{dx} - \frac{N_\theta}{a} = -P \quad \textbf{(Eq. 2.5.2)}$$

A second equilibrium equation can be written by summing the moments in the θ plane about the center of the shell element $\left(\frac{1}{2}\Delta x\right)$.

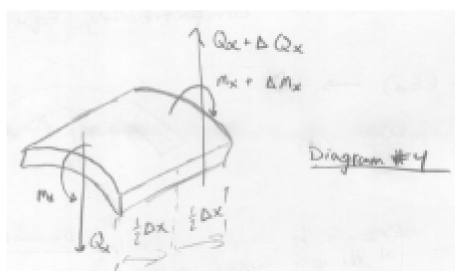

Figure 8. Differential Element, θ Plane Moment Reactions

Figure 8 shows only the terms inducing a moment in the θ plane. Writing out the equilibrium equation gives the following the following new expression.

$$\sum M_\theta = M_x - M_x - \Delta M_x + \frac{1}{2}\Delta x\,\Delta Q_x + \frac{1}{2}\Delta xQ_x = 0 \quad \textbf{(Eq. 2.5.3)}$$

In Equation 2.5.3 the $\frac{1}{2}\Delta x\Delta x$ term can be assumed to be small with respect to other terms and set to zero. Rearranging and dividing by Δx yields the second equilibrium equation.

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$$\frac{dM_x}{dx} = Q_x \quad \textbf{(Eq. 2.5.4)}$$

Equation 2.5.4 is identical to Equation 2.2.4, relating the bending moment to shear at any position *x* along the cylindrical shell. Substituting Equation 2.5.4 into Equation 2.5.2 pulls the shear component, Q_x out and yields the final equilibrium derived equation.

$$\frac{d^2M_x}{dx^2} - \frac{N_\theta}{a} = -P \quad \textbf{(Eq. 2.5.5)}$$

Next, through the use of compatibility and constitutive laws Equation 2.5.5 can be written in terms of the pipe's geometry and change in radial displacement, *w(x)*.

The first compatibility equation relates the circumferential strain to the radial displacement. Knowing that the change in radius is *w(x)*, and the original neutral plane radius is *a,* the circumferential strain, ϵ_θ can be written.

$$\epsilon_\theta = \frac{2\pi w(x)}{2\pi a} = \frac{w(x)}{a} \quad \textbf{(Eq. 2.5.6)}$$

Next, applying Hooke's Law and remembering N_θ has units of force per length, Equation 2.5.6 is used to write an expression for the circumferential resultant normal force.

$$N_\theta = \frac{Etw}{a} \quad \textbf{(Eq. 2.5.7)}$$

Equation 2.5.7 takes care of the second term in Equation 2.5.5 relating the cylindrical shell's internal circumferential resultant normal force to radial displacement. The next and final compatibility equation relates cylindrical shell's curvatures to its bending

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moments. The first goal is to obtain a relation for curvature in the θ direction, κ_θ. The pre-deformed θ curvature can be written as $\frac{1}{a}$, where *a,* is the radius of curvature. Next, the curvature at any point in time can be written as the following.

$$\kappa_\theta = \frac{1}{a+w(x)} = \frac{1}{a}\left(1 - \frac{w(x)}{a}\right) = -\frac{w(x)}{a^2} \quad \textbf{(Eq. 2.5.8)}$$

As stated in the previous section for this particular model we are assuming that w(x) is small with respect to the shell's original radius, therefore $\frac{w(x)}{a^2}$ is small and the change in curvature in the θ direction can be assumed to equal zero. The change in curvature in the theta direction is insignificant in comparison with the change in curvature in the x direction (Calladine, 1988).

The change in curvature in the x direction of the central neutral plane (z=0) can be written as Equation 2.2.10.

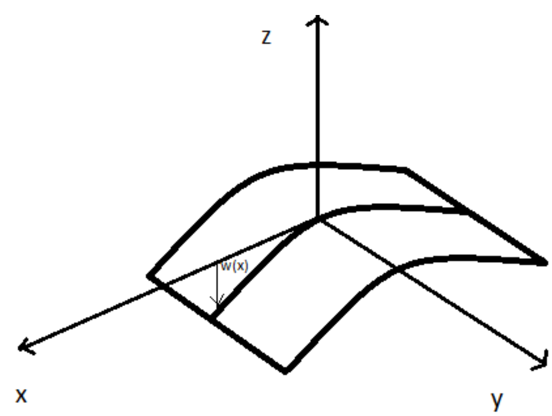

Figure 9. General Coordinate System and the z=0, Center Plane

For this specific derivation w(z) will resemble the change in plate shape from its original non-deformed shape, in the z direction, u(x) will resemble the change in shape in the x direction, and finally v(y) will resemble the change in shape in the y

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direction. Integrating both sides of Equation 2.2.10 and applying the natural boundary conditions, *w(x)* at *z* = 0 is written as

$$w(x) = -\frac{1}{2}\kappa_x x^2 \quad when \quad z = 0. \quad \textbf{(Eq. 2.5.9)}$$

Equation 2.5.9 is only accurate for regions close to the systems origin, and all strains due to bending are assumed to be small. This small curvature lets the central surface take a 'shallow' form, meaning the following can be assumed to be true (Calladine, 1988).

$$u(x) = v(y) = 0 \quad when\, z = 0 \quad \textbf{(Eq. 2.5.10)}$$

In order to write *w(x)* for the rest of plate element (various z values), Kirchhoff's hypothesis needs to be invoked. The hypothesis states that, "Through the point (x, y, 0) in the original configuration lay a normal to the central surface. This line in the material remains straight and normal to the central surface in the deformed configuration" (Calladine, 1988).

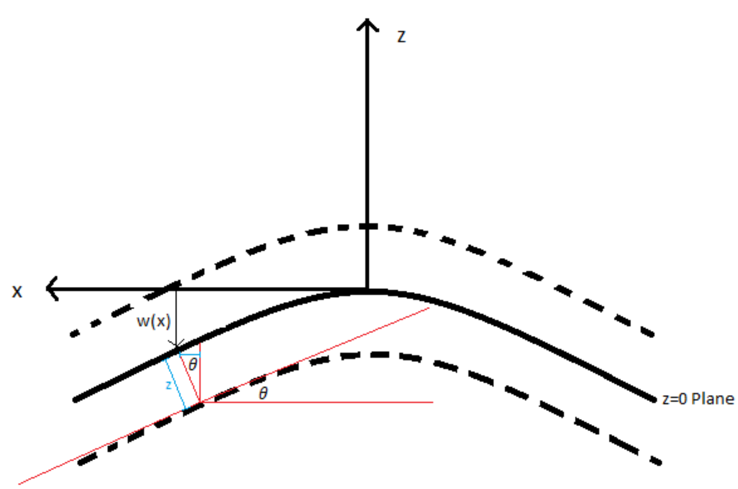

Figure 10. Kirchoff's Hypothesis

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Figure 10 shows the central plane with a zero slope at the origin, and corresponding normal surfaces lying above and below. From Equation 2.5.9 and Figure 10. Kirchoff's Hypothesis, it is clear that

$$\theta = \frac{dw}{dx}|_{z=0} = -x\kappa_x. \quad \textbf{(Eq. 2.5.11)}$$

Using Equation 2.5.11 along with small angle approximation the following expression for u(x) for any z can be written.

$$u(x) = z\sin[x\kappa_x] = zx\kappa_x \quad \textbf{(Eq. 2.5.12)}$$

Lastly using the strain displacement relationship the strain in the x direction can be written as the following.

$$\epsilon_x = \frac{du(x)}{dx} = z\kappa_x \quad \textbf{(Eq. 2.5.13)}$$

Applying Hooke's law for a plate element in one direction to the above strain yields the follow expression for the stress in the x-direction.

$$\sigma_x = \frac{E\varepsilon_x}{1-\nu^2} = \frac{Ez\kappa_x}{1-\nu^2} \quad \textbf{(Eq. 2.5.14)}$$

Taking moments about the center of the x-face and equating the external and internal forces by means of integration across the rectangular cross section (with thickness, *t*), results in the final moment curvature relationship for a shell in one direction.

$$M_x = \frac{Et^3\kappa_x}{12(1-\nu^2)} = D\kappa_x \quad \textbf{(Eq. 2.5.15)}$$

In this relationship, D is the flexural rigidity for a plate element. The flexural rigidity is analogous to the sectional modulus of the beam element (Calladine, 1988).

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Next, from the previous section Equation 2.2.10 can be used again to relate curvature in the x-direction to displacement, *w(x)*. Replacing the curvature with this expression yields the second compatibility equation needed to complete the derivation.

$$M_x = -D\frac{d^2w}{dx^2} \quad \textbf{(Eq. 2.5.16)}$$

Finally, substituting Equation 2.5.7 and 2.5.16 into Equation 2.5.5 yields the final governing ordinary differential equation describing the inward pinch of an externally pressured pipe.

$$D\frac{d^4w}{dx^4} + \frac{Et}{a^2}w = P \quad \textbf{(Eq. 2.5.17)}$$

Like Equation 2.2.12 from the earlier section, Equation 2.5.17 is a forth-order, constant coefficient, and again hosts *w(x)*, the outward change in radial displacement, as the dependent variable. The solution to Equation 1.5.17 has already been solved analytically. The differences in these two physical systems are their boundary conditions and constant coefficients values. It should be noted that each equation has different units, but are dimensionally consistent (Cook, 1985).

2.6 Cylindrical Shell Radial Displacement Analytical Solution

Having the same differential equation form, the variation in solution for cylindrical shells appears only in the β constant and particular solution. The β constant is written as

$$\beta = \left(\frac{Et}{4Da^2}\right)^{\frac{1}{4}}, \quad \textbf{(Eq.2.6.1)}$$

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and the particular solution becomes

$$w_p(x) = \frac{Pa^2}{Et}, \quad \textbf{(Eq.2.6.2)}$$

where P, in Equation 2.6.2 is the constant uniform external pressure. Boundary conditions have to be true for the entire circumference of the pipe. For example in order to use a fixed end natural boundary condition, the entire pipe needs to be fixed, but allowed to expand, which in most cases is true. Same is true for kinematic boundary conditions, if applied moment is applied to pipe end, it must be applied to the entire circumference of the pipe.

Once deflection *w(x)* is known, it is possible to back substitute and solve for the moment in the x direction, M_x, the transverse shear, Q_x, the circumferential normal stress, N_θ, and lastly, the curvature in the x-direction. Each deflection value relationship is reiterated below.

$$M_x = -D\frac{d^2w}{dx^2}, \quad Q_x = \frac{dM_x}{dx}, \quad N_\theta = \frac{Et}{a}w, \quad \kappa_x = \frac{M_x}{D} \quad \textbf{(Eq.2.6.3)}$$

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CHAPTER 3: Elastic Layer Coupled Beams and Shells

3.1 Generic Coupled Beam Problem

Coupled beam deflection theory is one methodology used to model simple electronic components laying on a circuit board. The electronic component itself is the elastic layer with some layer modulus, and the circuit board and component body are modeled as the coupled beams (Engel, 1993).

Using the same methodology from the previous chapter, it is possible to obtain a set of differential equations and their analytical solutions describing the static displacement of a coupled beam system.

3.2 Coupled Beam Derivation

The coupled beam differential equations take the same form as the single beam on an elastic foundation equation. The reaction force, which will now be called the interaction force (Pitarresi, Ceurter 1) coming from the elastic layer is dependent on the deflection of two beams and is present in both equations. Having this interaction force present in both beam equations allows the equations to be easily decoupled and analytically solved. The following figure shows general setup of the coupled beam system.

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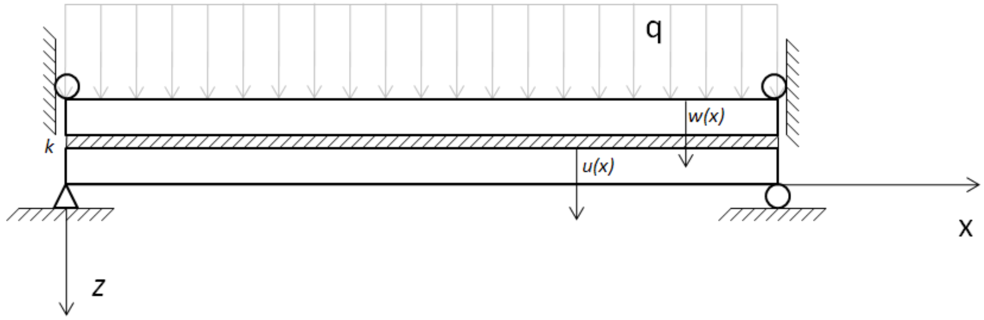

Figure 11. Coupled Beam System

Figure 11 shows two uniform beams with pinned boundary conditions at x equals zero and x equals L (boundary conditions being arbitrary). The top beam is acted on upon by a constant distributive load q (N/m). The beams are coupled together with an elastic layer which has an elastic modulus of k (N/m/m). This is true, assuming both beams have identical non-varying cross sections. The deflection of the top beam will be denoted as $w(x)$, and the deflection of the bottom beam, $u(x)$. The sectional modulus of the top and bottom beam will be referred to as, EI_{Top} and EI_{Bot}. Using the same methodology as the single beam, an infinitesimal piece is taken from both beams and the following free body diagrams with exposed reactions are formed.

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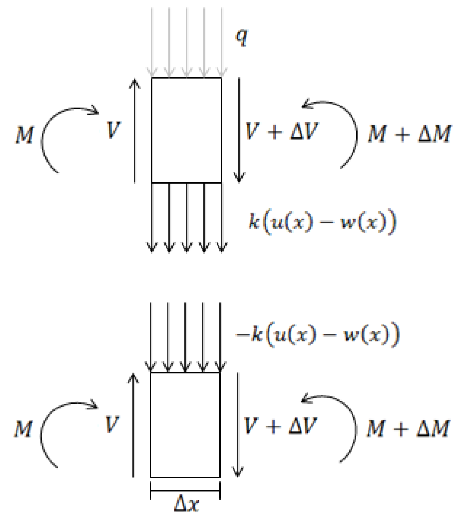

Figure 12. Top and Bottom Beam Infinitesimal Free Body Diagrams

Free body diagrams in Figure 12 are identical to the free body diagram in Figure 2, with the exception of distributed loads. The top beam is acted on by uniform distributed load q (N/m) and interaction force, which will be defined as the following.

$$Q = k\big(w(x) - u(x)\big) \quad \textbf{(Eq. 3.2.1)}$$

Invoking equilibrium to both free body diagrams, and applying compatibly and constitutive laws the following two equations can be written.

$$EI_{TOP}\frac{d^4w}{dx^4} + kw(x) - ku(x) = q \quad \textbf{(Eq. 3.2.2)}$$

$$EI_{BOT}\frac{d^4u}{dx^4} - kw(x) + ku(x) = 0 \quad \textbf{(Eq. 3.2.3)}$$

For sake of later comparison with the coupled cylindrical shell problem, Equation 3.2.2 and Equation 3.2.3 can be written in matrix form as the following.

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$$\begin{bmatrix} EI_{Top} & 0 \\ 0 & EI_{BOT} \end{bmatrix} \begin{bmatrix} w^{IV}(x) \\ u^{IV}(x) \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} w(x) \\ u(x) \end{bmatrix} = \begin{bmatrix} q \\ 0 \end{bmatrix} \quad \textbf{(Eq. 3.2.4)}$$

Equation 3.2.4 can easily be broken down into state space form (eight first order ODEs) and solved numerically. Alternatively, because Equation 3.2.2 and Equation 3.2.3 share common term Q (shown in Equation 3.2.1) the coupled equations can be solved analytically.

3.3 Analytical Coupled Beam Solution

The first step to analytically solve the coupled beam problem is to decouple Equation 3.2.4 and Equation 3.2.5 by substituting common term Q. Doing this yields,

$$w^{IV}(x) = \frac{q-Q}{EI_{TOP}} \quad \textbf{(Eq. 3.3.1)}$$

$$u^{IV}(x) = \frac{Q}{EI_{BOT}} \quad \textbf{(Eq. 3.3.2)}$$

where Q is given in Equation 3.2.3.

Taking the 4th derivative of both Equation 3.2.3 results in the following expression of 4th derivative terms $w^{IV}(x)$and $u^{IV}(x)$ and Q^{IV}.

$$Q^{IV} = k\left(w^{IV}(x) - u^{IV}(x)\right) \quad \textbf{(Eq. 3.3.3)}$$

Lastly substituting Equation 3.3.1 and Equation 3.3.2 into Equation 3.3.3 yields a single first order ordinary differential equation in terms of Q, the interaction force between the coupled beams.

$$Q^{IV} + Q\left(\frac{1}{EI_{TOP}} + \frac{1}{EI_{BOT}}\right)k = \frac{qk}{EI_{TOP}} \quad \textbf{(Eq. 3.3.4)}$$

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Now in order to obtain the analytical solution for the deflection of both beams, the expression for interaction force needs to be found by solving differential Equation 3.3.4. Once the interaction force is acquired, its expression can be substituted into Equations 3.3.1 and 3.3.2, which are now decoupled 4^{th} order differential equations with dependent variables *w(x)* and *u(x)*.

Equation 3.3.4 takes the same form as the single beam on elastic foundation differential Equation 2.2.12 shown in Chapter 2, with varying constants. Like Equation 2.2.12, a basic substitution can be made to help simplify the solution.

$$\boldsymbol{\lambda} = \left(\frac{\boldsymbol{k}}{\boldsymbol{4}}\left(\frac{\boldsymbol{1}}{\boldsymbol{EI_{TOP}}} + \frac{\boldsymbol{1}}{\boldsymbol{EI_{BOT}}}\right)\right)^{\frac{1}{4}} \quad \textbf{(Eq. 3.3.5)}$$

Solving differential Equation 3.3.4 for the interaction force Q results in the following expression (solving for both the homogeneous and particular solution together).

$$Q(x) = e^{\lambda x}(C_1 Sin[\lambda x] + C_2 Cos[\lambda x]) + e^{-\lambda x}(C_3 Sin[\lambda x] + C_4 Cos[\lambda x]) + \frac{EI_{BOT}q}{EI_{BOT}+EI_{TOP}} \quad \textbf{(Eq. 3.3.6)}$$

Knowing the relationship between the interaction force and deflections, w(x), and u(x) (Equation 3.2.3), allows to write interaction force boundary conditions from natural boundary conditions. Constants C_1 through C_4 are obtained using these boundary conditions. Now that an expression for Q has been determined, a back substitution can be made into differential Equations 3.3.1 and 3.3.2. The homogeneous solutions of the two equations take the form of a 3^{rd} order polynomials. The particular solution can be found by integrating Q(x) four times. Equation 3.3.4 shows the relationship between the forth-derivative of Q(x) and Q(x), making the

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integration trivial. Resulting equations for both the deflection of the top beam w(x), and bottom beam u(x), are shown below.

$$\boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{A_4} + \boldsymbol{A_3 x} + \boldsymbol{A_2}\frac{\boldsymbol{1}}{\boldsymbol{2}}\boldsymbol{x}^2 + \boldsymbol{A_1}\frac{\boldsymbol{1}}{\boldsymbol{6}}\boldsymbol{x}^3 + \frac{e^{\lambda x}}{\lambda^4 \boldsymbol{EI_{TOP}}}(C_1 Sin[\boldsymbol{\lambda} x] + C_2 Cos[\boldsymbol{\lambda} x]) + \frac{e^{-\lambda x}}{\lambda^4 \boldsymbol{EI_{TOP}}}(C_3 Sin[\boldsymbol{\lambda} x] + C_4 Cos[\boldsymbol{\lambda} x]) + \frac{1}{EI_{BOT}+EI_{TOP}}\left(\frac{EI_{BOT}q}{\lambda^4 \boldsymbol{EI_{TOP}}}\right) \quad \textbf{(Eq. 3.3.7)}$$

$$u(x) = \boldsymbol{B_4} + \boldsymbol{B_3 x} + \boldsymbol{B_2}\frac{\boldsymbol{1}}{\boldsymbol{2}}\boldsymbol{x}^2 + \boldsymbol{B_1}\frac{\boldsymbol{1}}{\boldsymbol{6}}\boldsymbol{x}^3 - \frac{e^{\lambda x}}{\lambda^4 \boldsymbol{EI_{BOT}}}(C_1 Sin[\boldsymbol{\lambda} x] + C_2 Cos[\boldsymbol{\lambda} x]) - \frac{e^{-\lambda x}}{\lambda^4 \boldsymbol{EI_{BOT}}}(C_3 Sin[\boldsymbol{\lambda} x] + C_4 Cos[\boldsymbol{\lambda} x]) + \frac{1}{EI_{BOT}+EI_{TOP}}\left(\frac{q}{\lambda^4}\right) \quad \textbf{(Eq. 3.3.8)}$$

Constants $\boldsymbol{A_1}$ through $\boldsymbol{A_4}$ and $\boldsymbol{B_1}$ through $\boldsymbol{B_4}$are found by applying the systems natural or kinematic boundary conditions. A specific case scenario with a finite element comparison is worked out in Chapter 3.

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3.4 Coupled Cylindrical Shell System Problem

A two or more coaxial cylindrical shell system, where individual shells are coupled by a linear elastic layer, is a unique unpublished problem. This simplistic shell system model may be used to represent two or more coaxial pipes with an intermediate variable elastic insulation undergoing an external pressure from something like a clamp, or water. Knowing the radial deflection of each pipe gives the modeler a general sense of failure criteria, and can be a rough initial start to a basic design. It is worth noting that modeled pipes are assumed to have a relatively small thickness with respect to their nominal diameter.

3.5 Coupled Cylindrical Shell System Derivation

In deriving the coupled shell problem, the same general approach can be taken as the coupled beam derivation. Infinitesimal free body diagrams of both the inner and outer shells are drawn. Equilibrium, constitutive, and compatibility laws are invoked forming the system of differential equations. Once the pinch equations are found, a relationship between the two shells can be distinguished. The elastic layer now reacts with pressure and has a elastic modulus with units of $\frac{N}{m^2}\Big/_{m}$.

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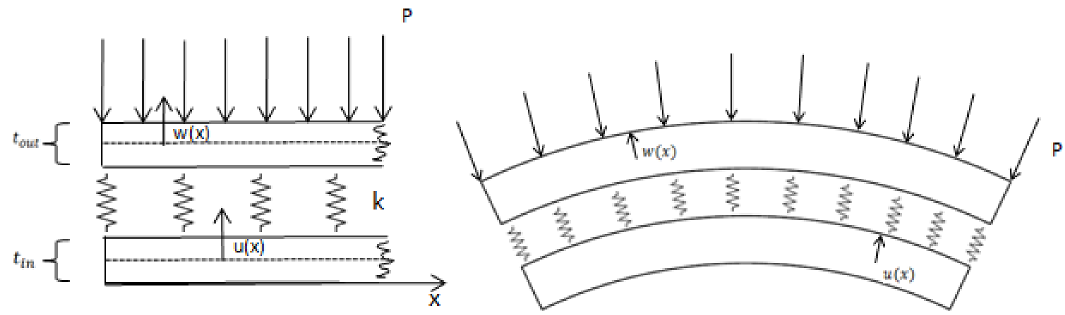

Figure 13. Coupled Cylindrical Shell System

Figure 13 shows the general coordinate system setup. Using the same direction as the single shell derivation, positive radial deflection of both shells is assumed to be coming out of plane. The outer radial deflection is denoted as *w(x)*, and the inner *u(x)*. Pressure P, $(\frac{N}{m^2})$, acts on the outer surface only. The elastic layer now exerts a pressure on the inside of the outer shell and the outside of the inner shell. Knowing this, the previously used cylindrical shell theory can be used in perform the derivation. Again, this theory assumes only a radial deflection, meaning no circumferential deformation; and the pressure induced from the intermediate elastic layer will always act normal to the shell's surfaces. The elastic modulus, k, has units of N/m^2/m. The outer and inner thickness will be denoted as t_{out} and t_{in}, flexural rigidities, D_{out} and D_{in}, and moduli of elasticity, E_{out} and E_{in}.

Knowing the elastic layer itself does not affect internal reactions of the cylindrical shells, and only acts externally on them with a pressure dependent on radial deflection, allows the exact same methodology to be used as the single shell derivation. Equation 2.5.17 can be used to write both coupled ordinary differential equations.

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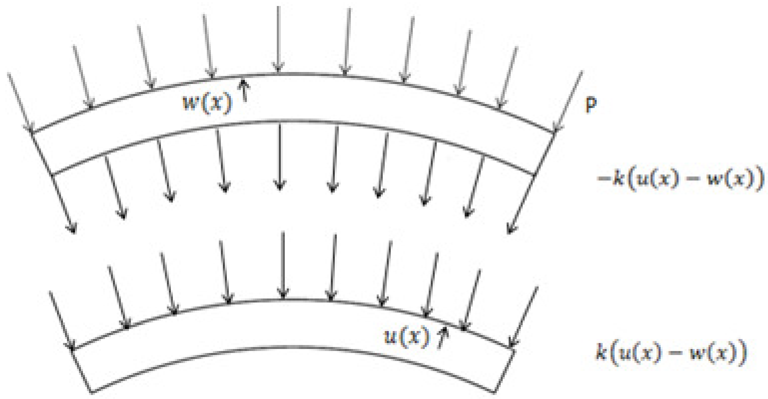

Figure 14. Outer and Inner Infinitesimal Free Body Diagrams

Figure 14 shows the pressure free body diagrams when an external pressure P acts on the outer shell surface. Intermediate pressure dependent on radial deflections w(x), and u(x) act equal and opposite. Using Equation 2.5.17 and knowing P is the summation of external pressures both coupled differential equations are written as the following.

$$D_{out}\frac{d^4w(x)}{dx^4}+\frac{E_{out}t_{out}}{a^2}w(x)=-P-kw(x)+ku(x) \qquad \textbf{(Eq. 3.5.1)}$$

$$D_{in}\frac{d^4u(x)}{dx^4}+\frac{E_{in}t_{in}}{b^2}u(x)=+kw(x)-ku(x) \qquad \textbf{(Eq. 3.5.2)}$$

The *a* and *b* variables in Equations 3.5.1 and 3.5.2 represent the outer and inner radii to the neutral axis of the cylindrical shells. As was done in the previous section coupled equations can be written in matrix form as the following.

$$\begin{bmatrix}\boldsymbol{D_{out}} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{D_{in}}\end{bmatrix}\begin{bmatrix}\boldsymbol{w^{IV}(x)}\\ \boldsymbol{u^{IV}(x)}\end{bmatrix}+\begin{bmatrix}\frac{E_{out}t_{out}}{a^2}+k & \boldsymbol{-k}\\ \boldsymbol{-k} & \frac{E_{in}t_{in}}{b^2}+k\end{bmatrix}\begin{bmatrix}\boldsymbol{w(x)}\\ \boldsymbol{u(x)}\end{bmatrix}=\begin{bmatrix}\boldsymbol{-P}\\ \boldsymbol{0}\end{bmatrix} \quad \textbf{(Eq. 3.5.3)}$$

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CHAPTER 4: Results

4.1 Beam on Elastic Foundation

In order to get a broad understanding of creating finite element models that incorporate a Winkler foundation, the simple beam on an elastic foundation problem was analyzed. When tackling the problem it was thought of to use either a 2-D planar element, or multiple 1-D Combine spring elements to act as the foundation.

The idea behind plane element model is the following. Its material Young's modulus could resemble, with manipulation, a foundation's Winkler foundation. After a brief discussion, it was decided that because the plane element would support shear deformation, it could possibly skew results, not allowing for an accurate representation of the linear Winkler foundation.

There is an analogy that Winker foundations can be thought of as a bed of linear springs. The idea is that, if enough springs are placed in a model, the representation of a foundation can be close to exact. Through experimentation it was found that by having a minimum of four springs per a half of wavelength, the springs will accurately model a foundation (Cook, 1985). In terms of finite element modeling, if the beam element nodes share the nodes of the spring elements, both elements can be used in conjunction.

The system in Figure 1 of Chapter 2 was solved with fix-fix boundary conditions, no distributed load q, and a 1000 N point force P, located in the center of the beam. The cross section assigned was a 50 mm x 50 mm square. Lastly the length of the beam was assigned to be 1000 mm.

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On an ANSYS model, a 275 N/mm spring was placed every 12.5 mm through the entire span of the beam. Looking back at Equation 2.3.4, this means that there are 93 springs per half a wave length. This condition meets Cook's allowable 4 spring limit, and the smearing is acceptable (Cook, 1985). If each spring's force is estimated as a distributed load meaning,

$$\boldsymbol{q = kw(x)/L} \quad \textbf{(Eq. 4.1.1)},$$

where L is the spacing in between the springs and q is the constant distributed load, then a corresponding foundation modulus can be estimated as

$$\boldsymbol{k_{foundation} = \frac{k_{spring}}{L}} \quad \textbf{(Eq. 4.1.2)}.$$

Using Equation 4.1.2, the foundation modulus of this particular system, to be used in the Analytical solution, is estimated to be 22 N/mm/mm.

The following figure shows the results to the Analytical solution yielding from Equation 2.3.4, and finite element ANSYS solution.

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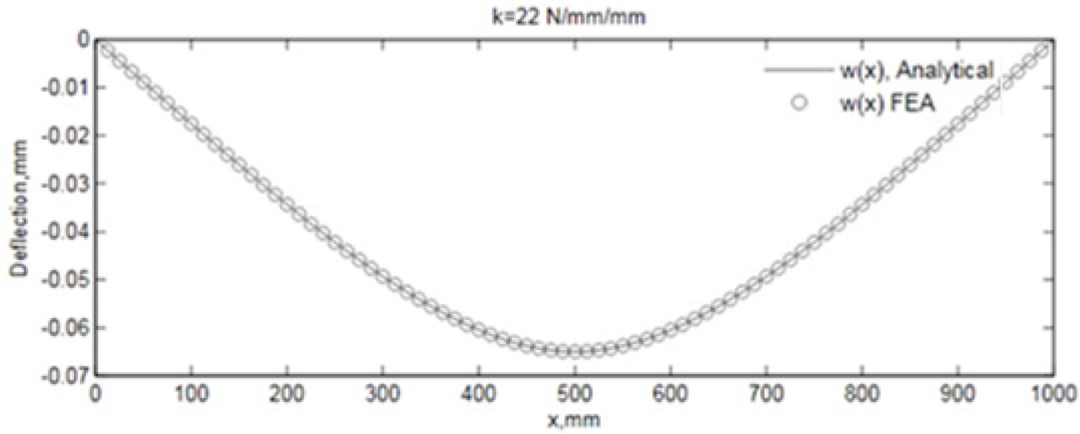

Figure 15. Pin-Pin Beam on an Elastic Foundation, FEA and Analytical Solution

Figure 15 verifies that a Winkler foundation can be estimated as a span of springs. Finite element data collection was obtained in a 12.5 mm interval. It is worth noting that the maximum displacement will always occur at the center of a pin-pin beam for a central point force loading. However, if a constant distributed loading is present, maximum deflection can be offset in both directions from the center. This will occur in the externally pressured pipe solution in the following section.

When the foundation modulus is set to zero, the $kw(x)$ term drops out Equation 2.2.12 and becomes the standard static, constant section modulus Euler-Bernoulli equation.

$$EI\,\frac{d^4w(x)}{dx^4} = q \quad \textbf{(Eq. 4.1.3)},$$

Equation 4.1.3 yields the small static deflection solution to a beam with a foundation modulus of zero.

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4.2 Cylindrical Shell With External Pressure

As learned previously, the beam on the elastic foundation static deflection problem yields the same solution as the cylindrical shell with a symmetric pressure loading radial deformation problem. In this section, a basic steel pipe's radial deflection will be calcuated analtically using Equation 2.5.17 and the coordinate system shown in Figure 4. Both ends of the pipes are constrained to allow no rotation and radial deflection; however, the pipe ends are allowed to deform in the x-direction.

A finite element analysis was done, again using ANSYS. Within ANSYS, the basic SHELL63 element was used to mesh the thin cylindrical shell. Within this four-node element, each node has six degrees of freedom, allowing translation and rotation in the x, y, and z direction. In order to assume equal end deformation in the x-direction, half the pipes length was taken, and a symmetry boundary condition was applied to one end of the pipe.

Chosen parameters for the problem are the following:

Constant	Definition	Value
L	Pipe Length	1000 mm
R	Nominal Radius	170 mm
t	Thickness	16 mm
E	Modulus of Elasticity	200,000 N/mm^2
ν	Poisson's Ratio	.333
P	Uniform Pressure	100 N/mm^2

Table 1. Cylindrical Shell with External Pressure Parameters

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This model roughly represents a 14-inch diameter pipe. Solving both analytically and by means of finite element analysis with the parameters shown above, the following solution is obtained.

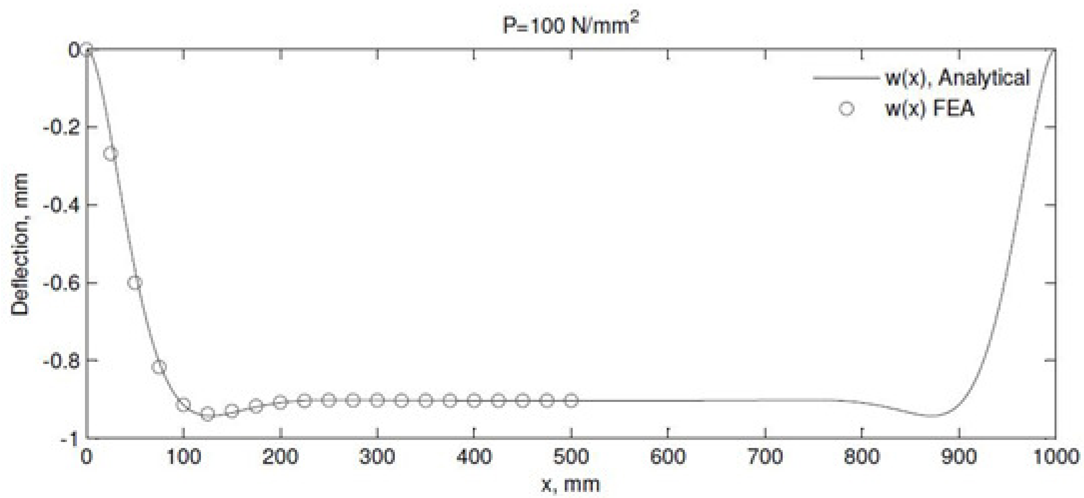

Figure 16. Cylindrical Shell FEA and Numerical Solution, P=100 N/mm^2

Figure 16 shows that the analytical solution checks with the finite element model (FEM). As stated above, the FEM utilized a symmetry boundary condition, explaining the half solution shown above. The shell element used to mesh the model will be an acceptable element to use when creating the coupled cylindrical shell model in section 4.4.

4.3 Coupled Beam

Figure 11 shows the general set up of the coupled beam case scenario. Both beams are equipped with pin-pin boundary conditions, and are coupled with a Winkler elastic layer.

The analytical solution to the coupled beam differential system Equation 3.2.4 was solved using Wolfram Mathematica. As stated in Chapter 3, the interaction force Q(x)

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is first found. Once the interaction force is obtained, through back substitution, the top beam deflection, *w(x)* and bottom beam deflection, u(x) are found.

The setup for finite element model for the coupled beam is very similar to the beam on the elastic foundation. For this case scenario the combine element springs are evenly distributed, attaching the parallel beams rather than just simply fixed to a ridge surface.

The spring spacing for the beam on the elastic foundation problem above was overly smeared, using more memory than was necessary to obtain an accurate solution. For this reason, the finite element coupled beam model was made using a larger spring spacing of 25 mm. Both beams have the same cross-section, material properties, and length as the beam on an elastic foundation case scenario. The stiffness of each spring was assigned to be 1000 N/mm. Using Equation 4.1.2, with the assigned spring spacing, the new elastic layer modulus becomes 40 N/mm/mm.

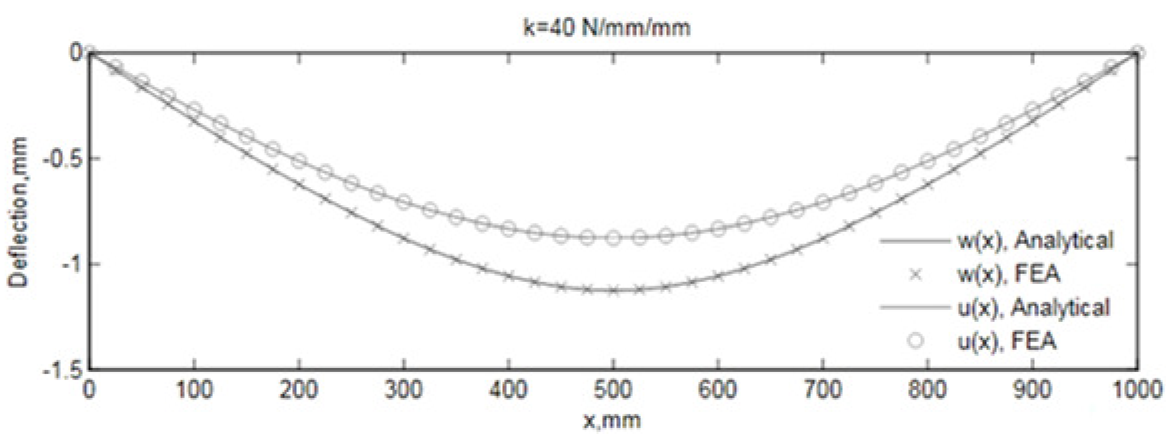

Figure 17. Pinned Coupled Beam, Point Load of 1000 N, FEA and Analytical Solution

Figure 17 shows the results to the static beam deflections of the coupled beam problem with central point load P, 1000 N on the top beam. As in the beam on

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an elastic foundation case scenario, the curve is smooth, and has a maximum deflection at the beams center, due to point load. If distributed loading was present a maximum deflection can occur symmetrically at an equal distance from the center of the beams.

In the special case scenario where the elastic layer modulus is set to zero, the equations in the system become decoupled, and the governing differential equations become simplified Euler-Bernoulli equations. For the case scenario above, this would mean that the top beam deflects with a 3^{rd} order polynomial shape, and the bottom beam has a zero deflection due to having zero external loading.

4.4 Cylindrical Shell Pipe System With External Pressure

Figure 13 shows the general cylindrical shell pipe system coordinate system setup (identical to single shell, with added elastic layer, and inner cylindrical shell). Both ends of the pipes are constrained to again, allow no rotation and radial deflection. However, both pipe are allowed to deform in the x-direction. To solve this problem, both a numeric solution to Equation 3.5.3 was obtained, and a FEM was built and solved.

Using the MATLAB finite difference function *bvp4c*, which simulates the three-stage Lobatto IIIa formula, Equation 3.5.3 was solved after being broken down into its eight 1-dimensional linear state-space differential equations. All numeric MATLAB functions and codes can be found in the appendix of this report.

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The FEM utilizes the methodologies used in both the coupled beam and the single cylindrical shell case scenarios. The shells in the system are swept areas, evenly meshed, and radially aligned with the SHELL63 element. For the systems elastic layer it was decided to again use the COMBIN14 spring element. The basic idea of spring placement is very similar to the coupled beam problem, but now each spring will be represented as a pressure, instead of a distributed load. Each 4-node shell element in the model is given the same size/area. Spring elements are directly attached to each shell element node of the inner cylinder, spanning to its corresponding radial shell element node of the outer cylinder. The following Figure shows both cylinders meshed elements.

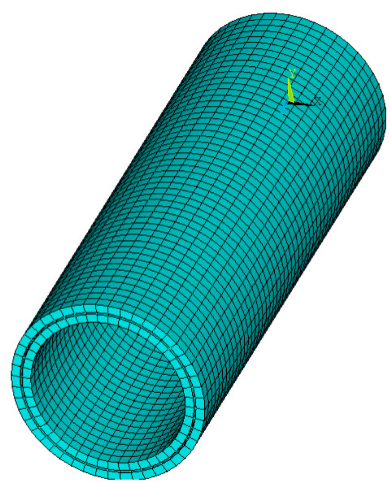

Figure 18. Cylindrical Shell System, FEA, Mesh

Using the same concept as Equation 4.1.1, each spring's force can be estimated as a pressure loading with the following definition

$$\boldsymbol{P = kw(x)/A}, \quad \textbf{(Eq. 4.1.1)}$$

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where $\boldsymbol{A}$ is the area between the springs, P is the estimated constant pressure loading, and k is the spring stiffness. From Equation 4.1.1, a corresponding foundation modulus can be estimated as

$$k_{foundation} = \frac{k_{spring}}{A}. \quad \textbf{(Eq. 4.1.2)}$$

In terms of the current case scenario, Area A is dependent on circumference therefore increasing as the radial length the spring is spanned. For the system above the spring spacing area is estimated at half the length of the springs.

Chosen geometry parameters, and material properties for the problem are the following:

Constant	Definition	Value
L	Pipe Length	1000 mm
a	Outer Nominal Radius	170 mm
b	Inner Nominal Radius	150 mm
t_{out}	Outer Thickness	16 mm
t_{in}	Inner Thickness	16 mm
E_{out}	Outer Modulus of Elasticity	200,000 N/mm^2
E_{in}	Inner Modulus of Elasticity	200,000 N/mm^2
ν_{out}	Poisson's Ratio	.333
ν_{in}	Poisson's Ratio	.333
P	Uniform Pressure	100 N/mm^2

Table 2. Cylindrical Shell Pipe System with External Pressure, Parameters

Using Equation 4.1.1, the foundation modulus to this particular FEM, to be used in the numerical calculation, was estimated to be 25 $N/mm/mm^2$. The following figure shows both the analytical and FEM solution for the radial pinch of both cylinders.

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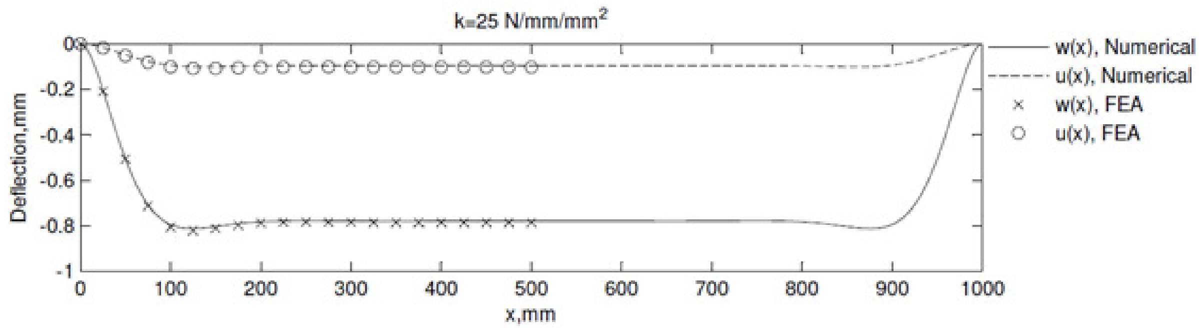

Figure 19. Coupled Cylindrical Shell FEA and Numerical Solution, k = 25 N/mm/mm^2

Figure 19 shows that the FEM solution is in strong agreement with the numerical solution. Both curves take the same general shape as the single shell problem.

Setting the elastic layer modulus/spring stiffness's to zero results in the decoupling of the governing system of differential equations. For a deflection to occur to either cylinder when the foundation modulus is set to zero, an external loading needs to be applied to the corresponding pipe. For the case scenario described above, only an external pressure was placed on the outside cylinder. The following figure shows the radial pinch of both cylinders when the foundation modulus is set to zero.

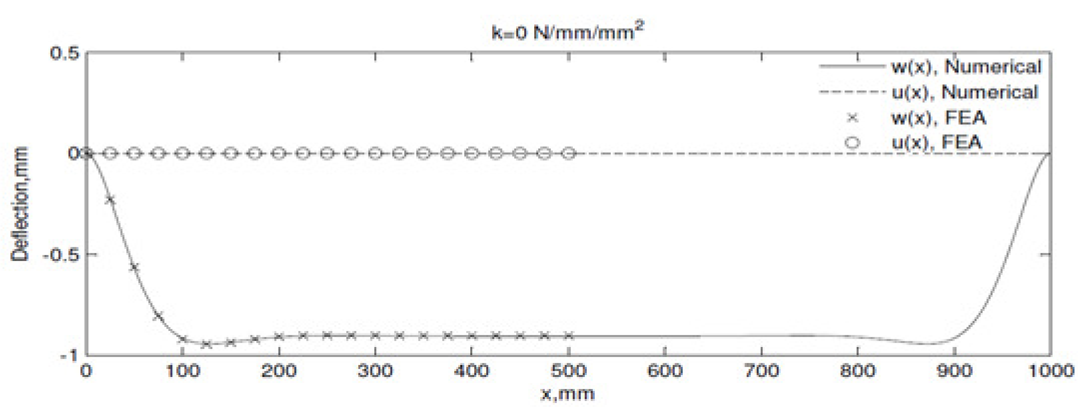

Figure 20. Coupled Cylindrical Shell FEA and Numerical Solution, k = 0 N/mm/mm2

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Figure 20 also shows a strong agreement between both solutions of the problem. As stated above, because the inner cylinder is not externally loaded, its radial pinch is zero. The outside shell acts like a single cylindrical shell with an external pressure loading of 100 N/mm^2. The outside shells geometry and boundaries are identical to the single shell case scenario in section 4.2. Therefore the radial deflection of the outer shell in Figure 20 is equal to the radial deflection of the single shell system in Figure 16.

One last analysis was done with the cylindrical shell system. If the elastic layer modulus in the system is set to infinite, in theory the springs should act like rigid bodies. This in return would cause both shells to deform an identical distance through the span of the cylinders. In Equation 3.5.3 if the elastic modulus is large the $\frac{E_{out}t_{out}}{a^2}$ and $\frac{E_{in}t_{in}}{b^2}$ terms are negligible. If both shells have the same material properties and equal thicknesses, the couple equations become identical. The following figure shows the MATLAB numerical solution to system Equation 3.5.3, with the above parameters, when the foundation modulus is set to a large value.

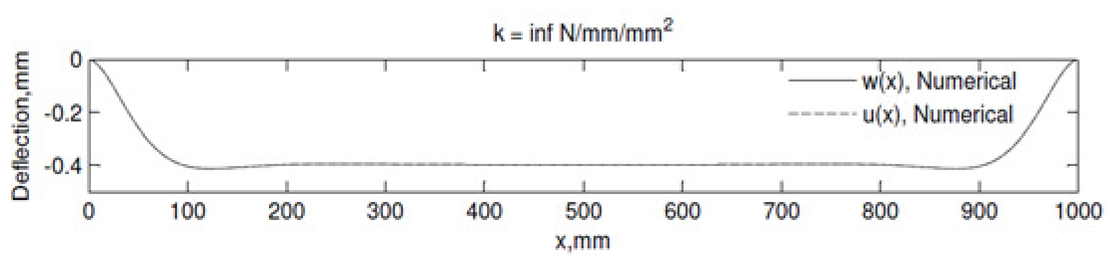

Figure 21. Coupled Cylindrical Shell FEA and Numerical Solution, Ridge Connections

Figure 21 confirms the identical radial deflection theory stated above. The shape of the plot conforms to the shapes of the other case scenarios.

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CHAPTER 5: Conclusion

5.1 Conclusion

It is well known that the governing differential equation describing the static deflection of a uniform beam on an elastic foundation takes the same general form as the equation for the static inward radial deflection of thin cylindrical shell. The derivations (differential calculus derived) and solution to these two ordinary differential equations were outlined in Chapter 2 of this Thesis. A finite element model (FEM) was built for two specific case scenarios verifying that the analytical solutions were valid. These two case scenarios also verified that the general setup (modeling construction, elements utilized, and applied loads) of the FEM were correct and could be again utilized when verifying later solutions.

The coupled beam static deflection problem, like the problems above, also has a well known solution. Its solution was first found by Suhir (1988) and further expanded on upon by Engel (1990), Caletka (1993), and Palmer (1991). The problem uses a linear Winkler elastic layer to couple two beams of identical cross sections. Differential calclus was used to derive its system of differential equations. The system is summarized in matrix form in Equation 3.2.4. The question introduced by this Thesis is, does the governing differential equation of the coupled beam problem match the equation of a coupled cylindrical shell problem.

The pressurized coupled cylindrical shell system remains unlooked at to this day. The system is composed of two thin coaxial cylindrical shells coupled with a linear

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Winkler elastic layer. Differential calculus was again utilized to obtain the problem's system of differential equations in Chapter 3 of this thesis. Equation 3.5.3 summarizes the system in matrix form.

Equation 3.2.4 governing the coupled beam problem is similar to Equation 3.5.3, but not exact like was found out for the single beam on an elastic foundation and single cylindrical shell problem. Added terms $\frac{E_{out}t_{out}}{a^2}$ and $\frac{E_{in}t_{in}}{b^2}$ have been added to the coefficient of the leading non-derivative dependent variable of each equation. These two terms dropping out of Equation 3.5.3 would lead to an identical set of differential equation matching the coupled beam problem. In a problem in which the cylindrical shell has a small Modulus of elasticity, small thickness, and large radius, these terms can be assumed to be zero. Also by setting k, the foundation modulus very large with respect to the terms above, the coupled equations become identical, decoupled, and have the same solution.

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5.2 Future Work

The coupled cylindrical shell system has been modeled and proven using a finite element analysis. The next step would be to design an experiment in which an actual pipe system is subjected to an external pressure and effectively measured. The experiment can be done using a variety of boundary conditions, loads, and elastic layer modulus.

Other work may involve finding the cylindrical shell system's non-dimensionalized analytical solutions. New formulations and relationships can be brought on upon by a dimensional analysis.

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APPENDIXES

A. Coupled Cylindrical Shell System Numerical Code

Program:MATLAB

File Name: pipesystemfunction.m

```
function dsdx = pipesystemfunction(x,s)
%Nolan Lott
%Declare State-Space Function pipesystemfunction.m
%Assigned Terms
%s(1)=w
%s(2)=w'
%s(3)=w''
%s(4)=w'''
%s(5)=u
%s(6)=u'
%s(7)=u''
%s(8)=u'''
%Declare Parameter Values
Eout=200*10^3;
Ein=200*10^3;
%k=.4681027738;
k=28;
%k=20;
%k=0
%k=10000000;
L=1000;
P=100;
Rout=170;
Rin=150;
v=.33;
tin=16;
tout=16;
Din=(Ein*tin^3)/(12*(1-v^2));
Dout=(Eout*tout^3)/(12*(1-v^2));
c1=((Eout*tout)/Rout^2)+k;
c2=((Ein*tin)/Rin^2)+k;
%List first order ODE in column vector
dsdx=zeros(8,1);
dsdx(1) = s(2);
dsdx(2) = s(3);
dsdx(3)=s(4);
dsdx(4)=(-P+k*s(5)-c1*s(1))/Dout;
dsdx(5)=s(6);
dsdx(6)=s(7);
dsdx(7)=s(8);
dsdx(8)=(k*s(1)-c2*s(5))/Din;
```


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Program: MATLAB

File Name: eightbc.m

```
function res = eightbc(sa,sb)
%Nolan Lott
%Declare Boundary Condition Function EIGHTBC.m
res=zeros(4:1);
res(1)=sa(1);
res(2)=sa(2);
res(3)=sb(1);
res(4)=sb(2);
res(5)=sa(5);
res(6)=sa(6);
res(7)=sb(5);
res(8)=sb(6);
```

Program: MATLAB

File Name: calc10_18_10.m

```
clear
clc
%Nolan Lott
%Assign Parameter
L=1000;
%Declare Intial Guesses
solinit = bvpinit(linspace(0,L,100),[1 0 1 0 1 0 1 0]);
%Use the three-stage Lobatto IIIa formula
sol = bvp4c(@pipesystemfunction,@eightbc,solinit);
x=linspace(0,L,1000);
y=deval(sol,x);
%Show Maximum Deflection
wmax=min(y(1,:))
umax=min(y(5,:))
xxx=linspace(0,500,21)
%Import FEA Solution
%UFEA=[] (21x1) (Import from excel)
%WFEA=[] (21x1) (Import from excel)
%Plot Numerical and FEA Solution
plot(x,y(1,:),x,y(5,:),'--',xxx,WFEA,'x',xxx,UFEA,'o')
daspect([1 .004 1])
title('k=25 N/mm/mm^2')
legend('w(x), Numerical','u(x), Numerical','w(x), FEA','u(x), FEA')
xlabel('x,mm')
ylabel('Deflection,mm')
```


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B. Coupled Cylindrical Shell System FEA Code

Program: ANSYS

```
/PMACRO
FINISH          !This prevents you from needing
            !to "finish" previous analyses or postprocesses
/CLEAR          !Clear memory of previous run

/filname,Cylindrical Shell Coupled System    ! Define the File Name

/title, Cylindrical Shell Coupled System, Nolan Lott

!*****************************************************************
!**                                                             **
!**                          ELEMENTS                           **
!**                                                             **
!*****************************************************************

/PREP7                  ! Activate Pre-processor

ET,1,SHELL63            ! Define element type 1

ET,2,COMBIN14
!*
*set, t, 16
*set, r1, 150
*set, r2, 170

*set, depth, 500
*set, NUMPOINTROUND_1A, 12*4
*set, NUMsegz, 25

WPLANE, , 0,0,
CSYS, WP,
!****************************************************************
!**                                                            **
!**                  MATERIAL AND REAL CONST                   **
!**             MATERIAL PROPERTIES OF POLY-SILICON            **
!**                                                            **
!****************************************************************

*set, youngs, 200e3
*set, poisson, 0.33
!Foundation Modulus = 25 N/mm^2/m
*set, k, 8377.58041

!!!! Define Material Properties
```


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```
UIMP, 1, EX, , , youngs,
UIMP, 1, NUXY, , , poisson,
UIMP, 1, DENS, , , density,
UIMP, 1, ALPX, , , CTE,

R,1,k, , , , , ,
R,2,t,t,t,t
R,3,k/2, , , , , ,
K, 1, 0,0
ALLSEL,ALL

CIRCLE,1,r1,,,,NUMPOINTROUND_1A
*GET, MAX_KP_0, KP, , NUM, MAX,
CIRCLE,1,r2,,,,NUMPOINTROUND_1A
*GET, MAX_KP_1, KP, , NUM, MAX,

!*************************************************************************
! Sweep function to generate stiffners

   allsel,all
            *get, kp_num_max, kp, , num, max,
   ! Define the path line
   k, kp_num_max+1, -1, 0, 0
   k, kp_num_max+2, -1, 0, -depth/NUMsegz
*get, line_num_max, line, , num, max
   l, kp_num_max+1, kp_num_max+2,
   *do, m, 1, line_num_max, 1
     adrag, m, , , , , , line_num_max+1
   *enddo

   !!!!! Delete the line and key points used for the sweep function
   ldele, line_num_max+1
   kdele, kp_num_max+1,
   kdele, kp_num_max+2,

agen, NUMsegz,ALL, , ,,,-depth/NUMsegz,, ,0

!allsel, all
!lsel, inve,
!*do, ii, 0, NUMPOINTROUND_1A-1, 1
!L,MAX_KP_0-ii,MAX_KP_1-ii
!*ENDDO
!lgen, NUMsegz+1,ALL, , ,,,-depth/NUMsegz,, ,0

!cm, skin1, line,

!allsel, all

nummrg, all
numcmp, all
```


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```
CSYS,0
!*********************************************************
!**                                                     **
!**                   MESHING & BC'S                    **
!**       AATT, MAT, REAL, TYPE, ESYS, SECN             **
!*********************************************************

aatt,1,2,1
allsel,all
ESIZE,0,1,
amesh,all
allsel, all
nummrg, all
numcmp, all

*do, ii2, 0, depth/NUMsegz+5, 1
*do, ii, 0, NUMPOINTROUND_1A-1, 1
CSYS,1
nsel,s,loc,y,0+360/(12*4)*ii
CSYS,0
nsel,u,loc,z,-1-depth/NUMsegz*ii2,-depth-depth
nsel,u,loc,z,1-depth/NUMsegz*ii2,depth+depth
*GET, MAX_KP_1, Node, , NUM, max,
*GET, Min_KP_1, Node, , NUM, min,
TYPE,2
MAT,1
REAL,1
ESYS,0

e,Min_KP_1,MAX_KP_1
*ENDDO
*ENDDO
EPLOT
/VIEW,1,1,1,1
/ANG,1

!!!!!!Animate
/REP,FAST
/DIST,1,1.08222638492,1
/REP,FAST
/DIST,1,1.08222638492,1
/REP,FAST
/DIST,1,1.08222638492,1
/REP,FAST
/DIST,1,1.08222638492,1
/REP,FAST
/DIST,1,0.924021086472,1
/REP,FAST
/DIST,1,0.924021086472,1
/REP,FAST
/DIST,1,0.924021086472,1
```


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```
/REP,FAST
/DIST,1,0.924021086472,1
/REP,FAST
/DIST,1,0.924021086472,1
/REP,FAST
/DIST,1,0.924021086472,1
/REP,FAST
/DIST,1,0.924021086472,1
/REP,FAST
/DIST,1,0.924021086472,1
/REP,FAST
/DIST,1,0.924021086472,1
/REP,FAST
R,3,k/2, , , , , ,
RMORE, ,
!*
/DIST,1,0.924021086472,1
/REP,FAST
/DIST,1,0.924021086472,1
/REP,FAST
/DIST,1,0.924021086472,1
/REP,FAST
/DIST,1,0.924021086472,1
/REP,FAST
/DIST,1,0.924021086472,1
/REP,FAST
/DIST,1,0.924021086472,1
/REP,FAST
/DIST,1,0.924021086472,1
/REP,FAST
ESEL,ALL
ESEL,S,TYPE,,2
EPLOT
ESEL,S,TYPE,,2
/VIEW,1,1
/ANG,1
/REP,FAST
/VIEW,1,1,1,1
/ANG,1
/REP,FAST
FLST,2,48,2,ORDE,2
FITEM,2,2401
FITEM,2,-2448
EMODIF,P51X,REAL,3,
```


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```
!***********************************************************
!**                                                       **
!**                         BC'S                          **
!**                                                       **
!***********************************************************
CSYS,1
lsel,a,loc,x,r2
CSYS,0
lsel,u,loc,z,-1,-depth
DL,all, ,SYMM

!!!! selecting inner shell
CSYS,0
lsel,s,loc,z,-depth
CSYS,1
lsel,u,loc,x,r2
lsel,u,loc,x,r1+1,r2
DL,all, ,UX,
DL,all, ,UY,
DL,all, ,ROTX,
DL,all, ,ROTy,
DL,all, ,ROTz,
!!! selecting outer
CSYS,0
lsel,s,loc,z,-depth
CSYS,1
lsel,u,loc,x,r1,r2-1
DL,all, ,UX,
DL,all, ,UY,
DL,all, ,ROTX,
DL,all, ,ROTy,
DL,all, ,ROTz,

!***********************************************************
!**                                                       **
!**                        LOADS                          **
!**                                                       **
!***********************************************************
asel,s,loc,x,r2
*set,Pres,100
SFA,all,1,PRES,Pres
allsel,all
FINISH
/SOL
/STATUS,SOLU
SOLVE
/POST1
```


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```
!*********************************************************
!**                                                    **
!**                   Plot Deflection                  **
!**                                                    **
!*********************************************************
/EFACET,1
PLNSOL, U,Y, 0,1.0
/DIST,1,1.08222638492,1
/REP,FAST
/DIST,1,1.08222638492,1
/REP,FAST
/DIST,1,1.08222638492,1
/REP,FAST
/DIST,1,1.08222638492,1
/REP,FAST
/DIST,1,1.08222638492,1
/REP,FAST
/DIST,1,1.08222638492,1
/REP,FAST
/DIST,1,1.08222638492,1
/REP,FAST
/DIST,1,1.08222638492,1
/REP,FAST
/DIST,1,1.08222638492,1
/REP,FAST
/DIST,1,1.08222638492,1
/REP,FAST
/VIEW,  1, -0.936534164293    , -0.461577056643E-01, -0.347524424062
/ANG,   1,   10.0844319947
/REPLO
/DIST,1,0.924021086472,1
/REP,SLOW
/DIST,1,0.924021086472,1
/REP,SLOW
FLST,2,2,8
FITEM,2,0,170,-500
FITEM,2,0,170,0
PATH,www,2,30,20,
PPATH,P51X
AVPRIN,0, ,
PDEF, ,U,Y,AVG
PLPATH,UY
PRPATH,UY
FINISH
```


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